Effect of Rotation on Thermal Instability in Couple-Stress Elastico-Viscous Fluid

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The thermal instability of a layer of a couple-stress fluid acted on by a uniform rotation is considered. Following the linearized stability theory and normal mode analysis, the dispersion relation is obtained. For stationary convection it is found that rotation has a stabilizing effect, whereas the couple-stress has both stabilizing and destabilizing effects. It is found that the presence of rotation introduces oscillatory modes in the system. A sufficient condition for the non-existence of overstability is also obtained.

Key words: Termal Instability; Couple-stress Elastico-viscous Fluid; Uniform Rotation.

1. Introduction

The thermal instability of a fluid layer heated from below plays an important role in geophysics, oceanography, atmospheric physics etc., and has been investigated by many authors, e. g. Bénard [1], Rayleigh [2], Jeffreys [3]. A detailed account of the theoretical and experimental studies of so called Bénard convection in Newtonian fluids has been given by Chandrasekhar [4]. The Boussinesq approximation, which states that the density can be treated as a constant in all terms of the equations of motion except the external force term has been used throughout. Vest and Arpaci [5] have studied the stability of a horizontal layer of Maxwell's viscoelastic fluid heated from below. Bhatia and Steiner [6] have considered the effect of a uniform rotation on the thermal instability of a viscoelastic fluid and have found that rotation has a destabilizing influence, in contrast to the stabilizing effect on a Newtonian fluid. R. C. Sharma [7] has studied the thermal instability of a layer of Oldroydian [8] fluid acted on by a uniform rotation and found that rotation has destabilizing and stabilizing effects under certain conditions, in contrast to a Maxwell fluid where the effect is destabilizing.

There are many elastico-viscous fluid that cannot be characterized by Maxwell's or Oldroyd's constitutive relations. One type of elastico-viscous fluids are couple-stress fluids. The theory of couple-stress fluids has been formulated by Stokes [9]. One of the applications of couple-stress fluids is the study of the mechanisms of lubrication of synovial joints. A human joint is a dynamically loaded bearing which has an articular cartilage as bearing and a synovial fluid as lubricant. When a fluid film is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee and hip joints are loaded bearing synovial joints of the human body, and these joints have a low friction coefficient and negligible wear.

The normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish. According to the theory of Stokes [9], couple-stresses appear in noticeable magnitudes in fluids with very large molecules. Since the long chain hyaluronic acid molecules are found as additives in synovial fluids, Walicki and Walicka [10] modeled the synovial fluid as a couple-stress fluid. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical implications. Practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The efficiency of the physiological joint lubrication is caused by several mechanisms. The synovial fluid is, due to its content of the hyaluronic acid, a fluid of high viscosity, near to a gel. A layer of such a fluid, heated from below under the action of rotation, may find applications in physiological processes. Goel et al. [11] have studied the hydromagnetic stability of an unbounded couple-stress binary fluid mixture under rotation with vertical temperature and concentration gradients. Sharma et al. [12] have considered a couple-stress fluid with suspended particles heated from below. They have found that for stationary convection, couple-stress has a stabilizing effect whereas suspended particles have a destabilizing effect. In another study, Sunil et al. [13] have considered a couple-stress fluid heated from below in a porous medium in the presence of a magnetic field and rotation, and have found that for stationary convection, rotation postpones the onset of convection. The magnetic field and couple-stress may hasten the onset of convection in the presence of rotation, while in absence of rotation they always postpone the onset of convection.

Keeping in mind the importance of non-Newtonian fluids, the present paper is devoted to the study of such couple-stress fluids heated from below in the presence of uniform rotation.

2. Formulation of the Problem and Dispersion Relation

Consider an infinite horizontal layer of a couple-stress fluid of depth d which is acted on by a uniform rotation $\boldsymbol{\Omega}(0,0,\Omega)$ and a gravity force $\boldsymbol{g}(0,0,-g)$. This layer is heated from below so that a steady temperature β $(=|\mathrm{d}T/\mathrm{d}z|)$ is maintained.

Let Γ_{ij}, τ_{ij}, e_{ij}, δ_{ij}, μ, μ', v_i and x_i denote the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, viscosity, couple-stress viscosity, velocity vector and position vector, respectively. The constitutive relations for the couple-stress fluids are

$$\begin{aligned}\Gamma_{ij} &= -p\delta_{ij}+\tau_{ij},\\ \tau_{ij} &= 2(\mu-\mu'\nabla^2)e_{ij},\\ e_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right).\end{aligned} \tag{1}$$

Let $\boldsymbol{q}(u,v,w)$, p, ρ, T, ν and χ are respectively the velocity, pressure, density, temperature, kinematic viscosity and thermal diffusivity. Then the momentum balance, mass balance and energy balance equations of the couple-stress fluid (Stokes [9], Chandrasekhar [4]) in the presence of rotation are

$$\begin{aligned}\frac{\partial \boldsymbol{q}}{\partial t}+(\boldsymbol{q}\cdot\nabla)\boldsymbol{q} &= -\frac{1}{\rho_0}\nabla p+\boldsymbol{g}\left(1+\frac{\delta\rho}{\rho_0}\right)\\ &\quad+\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\boldsymbol{q}+2(\boldsymbol{q}\times\vec{\Omega}),\end{aligned} \tag{2}$$

$$\nabla\cdot\vec{\boldsymbol{q}}=0, \tag{3}$$

$$\frac{\partial T}{\partial t}+(\boldsymbol{q}\cdot\nabla)T=\chi\nabla^2 T. \tag{4}$$

The equation of state is

$$\rho=\rho_0[1-\alpha(T-T_0)], \tag{5}$$

where ρ_0, T_0 are, respectively, the reference density and reference temperature at the lower boundary and α is the coefficient of thermal expansion.

Here we make the Boussinesq approximation under which the density changes may be disregarded in all terms in the equations of motion except the one in the external force. In the initial state the velocity, density, pressure and temperature at any point in the fluid are respectively given by

$$\boldsymbol{q}=(0,0,0),\ \rho=\rho(z),\ p=p(z) \text{ and } T=T(z). \tag{6}$$

Let $\boldsymbol{q}=(u,v,w)$, δp, $\delta\rho$ and θ denote, respectively, the perturbations in velocity $(0,0,0)$, pressure p, density ρ, and temperature T.

Then the linearized perturbation equations are

$$\begin{aligned}\frac{\partial \boldsymbol{q}}{\partial t} &= -\frac{1}{\rho_0}\nabla\delta p+\boldsymbol{g}\frac{\delta\rho}{\rho_0}\\ &\quad+\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\boldsymbol{q}+2(\boldsymbol{q}\times\boldsymbol{\Omega}),\end{aligned} \tag{7}$$

$$\nabla\cdot\boldsymbol{q}=0, \tag{8}$$

$$\frac{\partial\theta}{\partial t}+(\vec{\boldsymbol{q}}\cdot\nabla)T=\chi\nabla^2\theta. \tag{9}$$

Within the framework of the Boussinesq approximation, (7)–(9) becomes

$$\begin{aligned}&\frac{\partial}{\partial t}\nabla^2 w-g\alpha\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)+2\Omega\frac{\partial\zeta}{\partial z}\\ &\quad=\left[\nu-\frac{\mu'}{\rho_0}\nabla^2\right]\nabla^4 w,\end{aligned} \tag{10}$$

$$\frac{\partial\zeta}{\partial t}-2\Omega\frac{\partial w}{\partial z}=\left[\nu-\frac{\mu'}{\rho_0}\nabla^2\right]\nabla^2\zeta, \tag{11}$$

$$\left[\frac{\partial}{\partial t}-\chi\nabla^2\right]\theta=\beta w, \tag{12}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, and $\zeta=\partial v/\partial x-\partial u/\partial y$ denotes the z-component of the vorticity.

We now analyze the disturbances into normal modes, assuming that the perturbation quantities are of the form

$$[w,\theta,\zeta] = [W(z),\Theta(z),Z(z)]\exp(ik_x x + ik_y y + nt), \tag{13}$$

where k_x, k_y, are wave numbers along the x and y directions, respectively, $k^2 = k_x^2 + k_y^2$ and n is, in general, a complex constant.

Letting $a = kd$, $\sigma = nd^2/\nu$, $F = \mu'/\rho_0 d^2\nu$, $(x,y,z) \to (\hat{x}d,\hat{y}d,\hat{z}d)$ and $p_1 = \nu/\chi$, (10)–(12), using (13), give

$$\left[\sigma(D^2-a^2)W + \left(\frac{g\alpha d^2}{\nu}\right)a^2\Theta + \left(T_A^{1/2}d\right)DZ\right] = [1-F(D^2-a^2)](D^2-a^2)^2 W, \tag{14}$$

$$[\{1-F(D^2-a^2)\}(D^2-a^2)-\sigma]Z = -\frac{T_A^{1/2}}{d}DW, \tag{15}$$

$$[D^2-a^2-p_1\sigma]\Theta = -\left(\frac{\beta d^2}{\chi}\right)W, \tag{16}$$

where $T_A = 4\Omega^2 d^4/\nu^2$ denotes the Taylor number and $D = \mathrm{d}/\mathrm{d}\hat{z}$.

Eliminating Θ and Z in (14)–(16), we obtain

$$\sigma(D^2-a^2)[D^2-a^2-p_1\sigma][\{1-F(D^2-a^2)\}\{D^2-a^2\}-\sigma]W - Ra^2[\{1-F(D^2-a^2)\}\{D^2-a^2\}-\sigma]W \tag{17}$$
$$-T_A[D^2-a^2-p_1\sigma]D^2W = (D^2-a^2-p_1\sigma)[\{1-F(D^2-a^2)\}\{D^2-a^2\}-\sigma]\{1-F(D^2-a^2)\}\{D^2-a^2\}^2 W,$$

where $R = g\alpha\beta d^4/\nu\chi$ stands for the Rayleigh number.

We now assume that the fluid layer is confined between two free boundaries. The case is of artificial nature, but due to mathematical simplicity it enables us to show the effect of rotation on the couple-stress fluid analytically.

The appropriate boundary conditions, with respect to which (14)–(16) must be solved, are

$$W = D^2W = D^4W = 0, \quad \Theta = 0, \quad DZ = 0 \text{ at } \hat{z}=0 \text{ and } \hat{z}=1. \tag{18}$$

Dropping the caps for convenience and using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish on the boundaries, and hence the proper solution of (17) characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{19}$$

where W_0 is a constant.

Substituting (19) in (17), we obtain the dispersion relation

$$R_1 = \big\{(1+x+i\sigma_1 p_1)[\{1+F_1(1+x)\}(1+x)+i\sigma_1][1+F_1(1+x)][1+x]^2 + i\sigma_1(1+x)(1+x+i\sigma_1 p_1)$$
$$\cdot[\{1+F_1(1+x)\}(1+x)+i\sigma_1] + T_1(1+x+i\sigma_1 p_1)\big\}\big\{[\{1+F_1(1+x)\}(1+x)+i\sigma_1]x\big\}^{-1} \tag{20}$$

where

$$R_1 = \frac{R}{\pi^4}, \quad T_1 = \frac{T_A}{\pi^4}, \quad a^2 = \pi^2 x,$$
$$\frac{\sigma}{\pi^2} = i\sigma_1 \text{ and } F_1 = \pi^2 F.$$

3. The Stationary Convection

For the case of stationary convection, i. e. $\sigma = 0$, (20) reduces to

$$R_1 = \frac{(1+x)^3 + 2F_1(1+x)^4 + F_1^2(1+x)^5 + T_1}{x[1+F_1(1+x)]}. \tag{21}$$

In the absence of the couple-stress parameter, (21) reduces to

$$R_1 = \frac{(1+x)^3 + T_1}{x},$$

a result given by Chandrasekhar [4], Eqn. (130), p. 95.

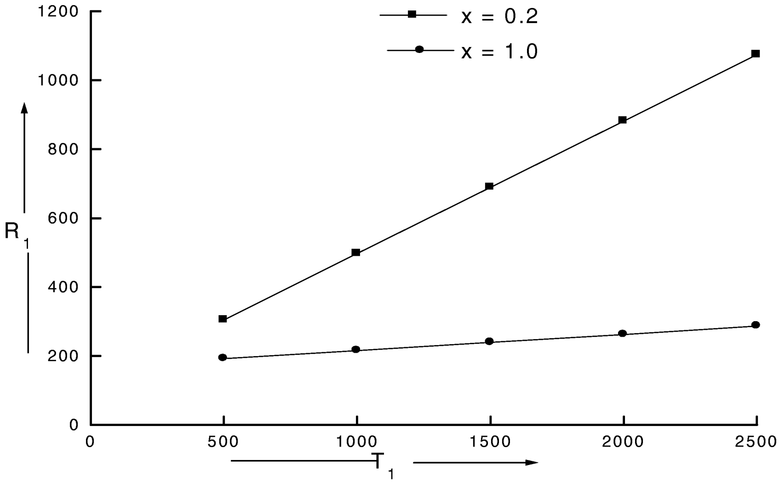

Fig. 1. The variation of R_1 with T_1 for fixed values of $F_1 = 10$ and $x = 0.2, 1.0$.

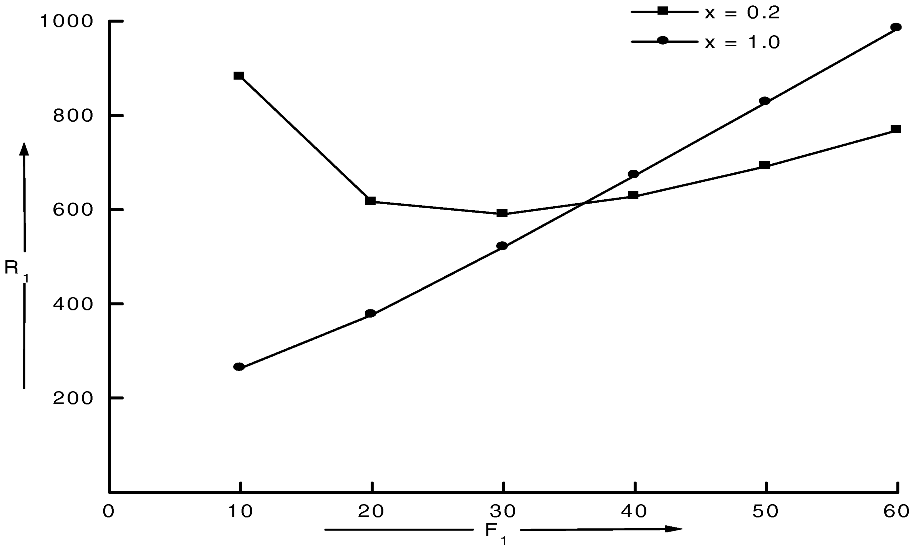

Fig. 2. The variation of R_1 with F_1 for fixed values of $T_1 = 2000$ and $x = 0.2, 1.0$.

To study the effect of rotation and the couple-stress parameter, we examine the nature of $\mathrm{d}R_1/\mathrm{d}T_1$ and $\mathrm{d}R_1/\mathrm{d}F_1$.

Equation (21) yields

$$\frac{\mathrm{d}R_1}{\mathrm{d}T_1} = \frac{1}{x[1+F_1(1+x)]}, \tag{22}$$

which is always positive. So the rotation has a stabilizing effect on the system.

$$\frac{\mathrm{d}R_1}{\mathrm{d}F_1} = \frac{1+x}{x}\left[2(1+x)^3 - \frac{\{(1+x)^3+2F_1(1+x)^4+F_1^2(1+x)^5+T_1\}}{[1+F_1(1+x)]^2}\right], \tag{23}$$

which may be positive as well as negative. The couple-stress parameter thus has both stabilizing and destabilizing effects on the system.

The dispersion relation (21) is also analysed numerically. Figure 1 shows the variation of R_1 with respect to T_1, for $F_1 = 10$ and the wave numbers $x = 0.2$, $x = 1.0$. The Rayleigh number R_1 increases with increase of the rotation parameter T_1, showing its stabilizing effect on the system. Figure 2 shows the variation of R_1 with respect to F_1, for fixed values of $T_1 = 2000$ and the wave numbers $x = 0.2$, $x = 1.0$. It depicts both the stabilizing and destabilizing effect of the couple-stress parameter on the system.

4. Stability of the System and Oscillatory Modes

Multiplying (14) by W^*, the complex conjugate of W, integrating over the range of z and making use of (15) and (16) together with the boundary conditions (18), we obtain

$$-\sigma I_1 + \frac{g\alpha\chi a^2}{\nu\beta}(I_2 + p_1\sigma^* I_3) - d^2(I_4 + FI_5) - d^2\sigma^* I_6 = I_7 + FI_8, \quad (24)$$

where

$$I_1 = \int_0^1 (|DW|^2 + a^2|W|^2)\mathrm{d}z,\ I_2 = \int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)\mathrm{d}z,$$

$$I_3 = \int_0^1 (|\Theta|^2)\mathrm{d}z, \quad I_4 = \int_0^1 (|DZ|^2 + a^2|Z|^2)\mathrm{d}z,$$

$$I_5 = \int_0^1 (|D^2Z|^2 + 2a^2|DZ|^2 + a^4|Z|^2)\mathrm{d}z,\ I_6 = \int_0^1 (|Z|^2)\mathrm{d}z,$$

$$I_7 = \int_0^1 (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)\mathrm{d}z, \quad (25)$$

$$I_8 = \int_0^1 (|D^3W|^2 + 3a^2|D^2W|^2 + 3a^4|DW|^2 + a^6|W|^2)\mathrm{d}z,$$

and σ^* is the complex conjugate of σ. The integrals $I_1 - I_8$ are all positive definite. Putting $\sigma = \sigma_r + i\sigma_i$ in (24) and equating the real and imaginary parts, we obtain

$$\sigma_r\left[-I_1 + \frac{g\alpha\chi a^2}{\nu\beta}p_1 I_3 - d^2 I_6\right] = -\frac{g\alpha\chi a^2}{\nu\beta}I_2 + d^2(I_4 + FI_5) + I_7 + FI_8, \quad (26)$$

and

$$\sigma_i\left[I_1 + \frac{g\alpha\chi a^2}{\nu\beta}p_1 I_3 - d^2 I_6\right] = 0. \quad (27)$$

In the absence of rotation, (27) reduces to

$$\sigma_i\left[I_1 + \frac{g\alpha\chi a^2}{\nu\beta}p_1 I_3\right] = 0, \quad (28)$$

and the terms in brackets are positive definite. Thus $\sigma_i = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied in the absence of rotation. It is clear from (27) that the presence of rotation brings oscillatory modes (as σ_i may not be zero) which were non-existent in their absence for a couple-stress fluid layer heated from below.

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